# Random Vector Norms

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# Norms

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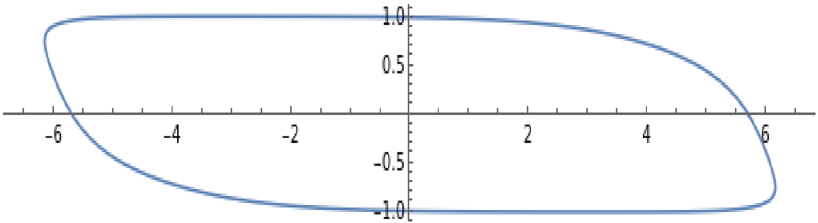
## In Other Words...

To prove that $\|\cdot\|$ is a norm, we must show three things:

- $\|\mathbf{a}\|$ equals zero when $\mathbf{a} = \mathbf{0}$ and is otherwise positive.
- $\|c\mathbf{a}\| = |c|\|\mathbf{a}\|$ for any scalar $c$.
- $\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$ for all $\mathbf{a}$, $\mathbf{b}$.

# Norm Unit "Circles"

A norm $\|\cdot\|$ is uniquely defined by the set $\{\mathbf{x} : \|\mathbf{x}\| = 1\}$, which can be any symmetric closed curve whose interior is convex and includes the origin.



# Euclidean Norm in $\mathbb{R}^2$

## Example

The Euclidean norm $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}_+$ is the familiar measure of the magnitude of a vector. If $\mathbf{x} = (x_1, x_2)$, then

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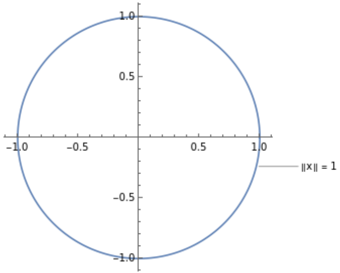

# $p$-Norms in $\mathbb{R}^2$

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The $p$ norm on $\mathbb{R}^2$ is the function $\|\cdot\|_p : \mathbb{R}^2 \to \mathbb{R}_+$ defined by

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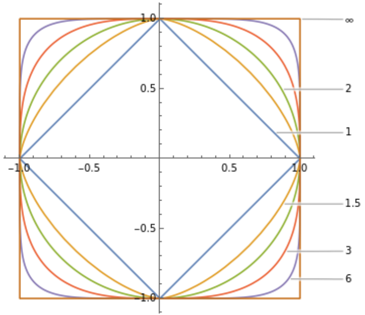

# Random Variables

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## Example

Let $X$ be a Bernoulli random variable with parameter $p$. Then $X$ is 1 with probability $P(1) = p$ and 0 with probability $P(0) = 1 - p$.

# Moments

## Definition

*The $d^{th}$ **moment** $\mu_d(X)$ of a random variable $X$ is the expected value of $X^d$.*

- $\mu_1 = \mathbf{E}[X]$ *is the mean.*
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## Example

Let $X$ be a Bernoulli random variable with parameter $p$. Then

$$\mu_d(X) = \mathbf{E}\left[X^d\right] = 0^d(1-p) + 1^d(p) = p.$$

# Random Vectors

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## Definition

*When all* $X_i$ *in a random vector* $\mathbf{X}$ *have the same distribution and are independent of each other,* $\mathbf{X}$ *is **independent and identically distributed**, or **i.i.d.***

# Random Vector Norms

## Definition

*Let $\mathbf{Y}$ be a random vector such that no $Y_i$ is a linear combination of other entries of $\mathbf{Y}$. Then the function*

$$\|\mathbf{x}\|_{\mathbf{Y},d} = \left(\mathbf{E}\,|\langle \mathbf{Y}, \mathbf{x}\rangle|^d\right)^{1/d}$$

*is a norm on $\mathbb{R}^n$. Here $\mathbf{x} \in \mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ is the dot product.*

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## Example

In $\mathbb{R}^n$,

$$\|\mathbf{x}\|_{\mathbf{Y},d} = \left(\mathbf{E}\,|Y_1 x_1 + Y_2 x_2|^d\right)^{1/d}$$

# Bernoulli Random Variables

Let $\mathbf{Y} = (Y_1, Y_2)$, where $Y_1$ and $Y_2$ are independent Bernoulli random variables with parameter $p$. Then $Y_1$ and $Y_2$ are each 1 with probability $p$ and 0 with probability $1 - p$.

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We calculate the norm by summing over all four combinations.

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$$\begin{aligned} \|\mathbf{x}\|_{\mathbf{Y},d}^{d} &= \mathbf{E}\Big|Y_1x_1 + Y_2x_2\Big|^d \\ &= p^2\Big|(1)x_1 + (1)x_2\Big|^d + (1-p)^2\Big|(0)x_1 + (0)x_2\Big|^d \\ &\quad + p(1-p)\Big|(1)x_1 + (0)x_2\Big|^d + (1-p)p\Big|(0)x_1 + (1)x_2\Big|^d \end{aligned}$$

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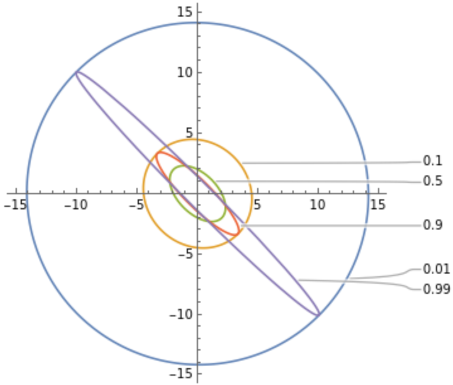
&= p^2\Big|x_1 + x_2\Big|^d + p(1-p)\left(|x_1|^d + |x_2|^d\right).
\end{aligned}$$

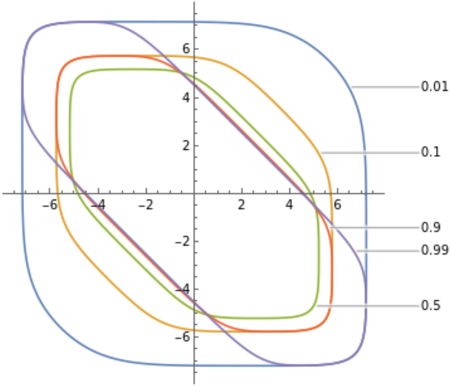
# Bernoulli Unit Circles

$$\|\mathbf{X}\|^2_{\mathbf{Y},2} = p^2|x_1 + x_2|^2 + p(1-p)\left(|x_1|^2 + |x_2|^2\right).$$



# Bernoulli Unit Circles

$$\|\mathbf{X}\|_{\mathbf{Y},10}^{10} = p^2|x_1 + x_2|^{10} + p(1-p)\left(|x_1|^{10} + |x_2|^{10}\right).$$



# Application

## Portfolio Optimization

- Let $\mathbf{Y}$ be a random vector where each $Y_i$ is a different investment distributed according to possible one-year returns, with mean 0.
- Let $\mathbf{x}$ be a vector of allocations, where each $x_i$ is the amount invested in the investment $Y_i$, and the sum of the $x_i$ is the total amount to be invested.

Then the standard deviation of the portfolio is

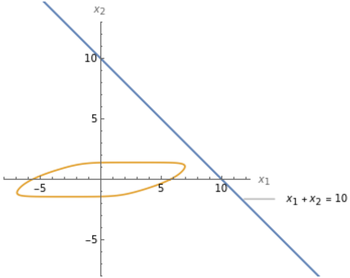
$$\|\mathbf{x}\|_{\mathbf{Y},2} = \left(\mathbf{E}|\langle \mathbf{Y}, \mathbf{x}\rangle|^2\right)^{1/2}.$$

# Portfolio Optimization

Since $\|\cdot\|_{\mathbf{Y},2}$ is a norm, it satisfies the triangle inequality, so any two risk-minimizing allocations have risk-minimizing average.

## Example

Suppose we have \$10. If $\|\cdot\|_{\mathbf{Y},2}$ has the unit circle below, the risk-minimizing allocation is to put \$8.5 into $Y_1$ and \$1.5 into $Y_2$.



# Symmetric Norms

## Definition

A norm $\|\cdot\|$ on $\mathbb{R}^n$ is *symmetric* if $\|\mathbf{x}\| = \|P\mathbf{x}\|$ for any $\mathbf{x} \in \mathbb{R}^n$ and permutation matrix $P$.

## Example

In $\mathbb{R}^2$, symmetry means that $\|(x_1, x_2)\| = \|(x_2, x_1)\|$.

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## Example

The $p$-norms $\|\mathbf{x}\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$ are symmetric.

# Hermitian Matrices

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The following matrix is Hermitian.

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## Proposition

Hermitian matrices have only real eigenvalues.

# Generalization to Hermitian Matrices

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*Let* $\|\cdot\|$ *be a symmetric norm on* $\mathbb{R}^n$. *Then* $\|A\|_{\mathrm{H}} = \|\boldsymbol{\lambda}(A)\|$ *is a norm on* $\mathrm{H}_n$.

## Corollary

*Let* $\mathbf{Y}$ *be i.i.d. Then* $\|\mathbf{x}\|_{\mathbf{Y},d} = \left(\mathbf{E}\,|\langle \mathbf{Y}, \mathbf{x}\rangle|^d\right)^{1/d}$ *is symmetric. Hence,* $\|A\|_{\mathbf{Y},d}^{\mathrm{H}} = \left(\mathbf{E}\,|\langle \mathbf{Y}, \boldsymbol{\lambda}(A)\rangle|^d\right)^{1/d}$ *is a norm on* $\mathrm{H}_n$.

# Bibliography

Ángel Chávez, Stephan Ramon Garcia, and Jackson Hurley.
Norms on complex matrices induced by random vectors.
*Canadian Mathematical Bulletin*, page 1–18, 2023. To appear.

# Thank you

Thank you, Professor Garcia, Professor Chávez, and all my math professors and peers!